Optimal feedback control simulation of a visuomotor rotation

Alexis Berland[1,2,3], Youssouf Ismail Cherifi[2], Alexis Paljic[2], Emmanuel Guigon[3]

[1]Higher Institute of Psychomotor Therapy, ISRP, Paris, France
[2]Mines Paris, PSL Research University, Centre for Robotics, CAOR, Paris, France
[3]Sorbonne Université, CNRS, Institut des Systèmes Intelligents et de Robotique, ISIR, F-75005 Paris, France

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1 General problem

We consider the following linear optimal feedback control problem: Find a control law $\boldsymbol{u}(t)$ ($t \in [t_0, t_f]$) such that the performance index

$$\mathcal{J}(t) = \frac{1}{2}\int_t^{t_f} ||\boldsymbol{u}(\sigma)||^2 d\sigma$$

is minimum, and an object with linear dynamics

$$\dot{\boldsymbol{x}}(t) = \mathbf{A}\boldsymbol{x}(t) + \mathbf{B}\boldsymbol{u}(t) + \boldsymbol{n}_{\text{dyn}} \tag{1}$$

evolves from state $\boldsymbol{x}(t) = \hat{\boldsymbol{x}}(t)$ to state $\boldsymbol{x}(t_f) = \boldsymbol{x}_f$, $\mathbf{A}$ is the dynamic matrix, $\mathbf{B}$ the control matrix, with the observation process

$$\boldsymbol{y}(t) = \mathbf{H}\boldsymbol{x}(t) + \boldsymbol{n}_{\text{obs}}$$

and the estimation process

$$\dot{\hat{\boldsymbol{x}}}(t) = \mathbf{A}\boldsymbol{x}(t) + \mathbf{B}\boldsymbol{u}(t) + \mathbf{K}(t)(\boldsymbol{y}(t) - \hat{\mathbf{H}}\hat{\boldsymbol{x}}(t)) \tag{2}$$

where $\boldsymbol{n}_{\text{dyn}}$ is noise on the dynamics, $\boldsymbol{n}_{\text{obs}}$ noise on the observation, $\mathbf{H}$ is the observation matrix, $\hat{\mathbf{H}}$ the internal representation of the observation matrix, and $\mathbf{K}$ the Kalman gain.

2 General solution: Kalman gain

The solution to this problem is the control law $\boldsymbol{u}(t)$ and the Kalman gain $\mathbf{K}(t)$ ($t \in [t_0, t_f]$). The way to find the control law is described in detail below in the case of a 2D linear mass. The general solution for the Kalman gain is the following. We discretize the object dynamics (Eq. 1) with the timestep η

$$\boldsymbol{x}_{\mathrm{t}+1} = \mathtt{A}\boldsymbol{x}_{\mathrm{t}} + \mathtt{B}\boldsymbol{u}_{\mathrm{t}} \tag{3}$$

for $\mathrm{t} = 1, \ldots, \mathrm{N}-1$ ($\mathrm{N} = t_f/\eta$). The Kalman gain is given by

$$\begin{cases} \mathbf{K}_{\mathrm{t}} = \mathtt{A}\mathbf{P}_{\mathrm{t}}\hat{\mathbf{H}}(\hat{\mathbf{H}}\mathbf{P}_{\mathrm{t}}\hat{\mathbf{H}}^T + \Omega_{\mathrm{obs}})^{-1} \\ \mathbf{P}_{\mathrm{t}+1} = \mathtt{A}\mathbf{P}_{\mathrm{t}}\mathtt{A}^T + \Omega_{\mathrm{dyn}} - \mathbf{K}_{\mathrm{t}}(\hat{\mathbf{H}}\mathbf{P}_{\mathrm{t}}\hat{\mathbf{H}}^T + \Omega_{\mathrm{obs}})\mathbf{K}_{\mathrm{t}}^T \end{cases} \tag{4}$$

where $\mathbf{P}_{\mathrm{t}}$ is the state covariance matrix, Ω_{dyn} the covariance matrix of noise on the dynamics and Ω_{obs} the covariance matrix of noise on the observation.

3 Specific solution: Control law

We consider a 2D mass actuated by second-order linear muscles. The mass+muscles dynamics is

$$\begin{cases} \mathrm{m}\ddot{X}(t) = a_X(t) \\ \mathrm{m}\ddot{Y}(t) = a_Y(t) \\ \tau\dot{a}_X(t) = -a_X(t) + e_X(t) \\ \tau\dot{a}_Y(t) = -a_Y(t) + e_Y(t) \\ \tau\dot{e}_X(t) = -e_X(t) + u_X(t) \\ \tau\dot{e}_Y(t) = -e_Y(t) + u_Y(t) \end{cases}$$

where (X,Y) is the mass position, (a_X,a_Y) is the muscles' activation, (e_X,e_Y) the muscles' excitation, (u_X,u_Y) the muscles' control, and τ the muscle time constant. The goal is to drive the mass from state $X(t_0) = X_0$, $Y(t_0) = Y_0$, $\dot{X}(t_0) = 0$, $\dot{Y}(t_0) = 0$, $a_X(t_0) = 0$, $a_Y(t_0) = 0$, $e_X(t_0) = 0$, $e_Y(t_0) = 0$ to state $X(t_f) = X_f$, $Y(t_f) = Y_f$, $\dot{X}(t_f) = 0$, $\dot{Y}(t_f) = 0$, $a_X(t_f) = 0$, $a_Y(t_f) = 0$, $e_X(t_f) = 0$, $e_Y(t_f) = 0$ with minimum cost. The performance index is

$$\mathcal{J} = \frac{1}{2}\int_{t_0}^{t_f} (u_X^2(\sigma) + u_Y^2(\sigma))d\sigma$$

We define x_0 as the ongoing cost, x_1, x_2 the position, x_3, x_4 the velocity, x_5, x_6 the muscle activation, and x_7, x_8 the muscle excitation. The dynamics is

$$\begin{cases} \dot{x}_0(t) = (u_1^2(t) + u_2^2(t))/2 \\ \dot{x}_1(t) = x_3(t) \\ \dot{x}_2(t) = x_4(t) \\ \mathrm{m}\dot{x}_3(t) = x_5(t) \\ \mathrm{m}\dot{x}_4(t) = x_6(t) \\ \tau\dot{x}_5(t) = -x_5(t) + x_7(t) \\ \tau\dot{x}_6(t) = -x_6(t) + x_8(t) \\ \tau\dot{x}_7(t) = -x_7(t) + u_1(t) \\ \tau\dot{x}_8(t) = -x_8(t) + u_2(t) \end{cases} \tag{5}$$

The Hamiltonian is

$$\begin{aligned} \mathcal{H} \quad = \quad & \tfrac{1}{2}\lambda_0(u_1^2 + u_2^2) + \lambda_1 x_3 + \lambda_2 x_4 + \lambda_3 x_5/\mathrm{m} + \lambda_4 x_6/\mathrm{m} + \lambda_5(-x_5 + x_7)/\tau + \\ & \lambda_6(-x_6 + x_8)/\tau + \lambda_7(-x_7 + u_1)/\tau + \lambda_8(-x_8 + u_2)/\tau \end{aligned}$$

Optimum is defined by

$$\begin{cases} \frac{\partial \mathcal{H}}{\partial u_1} = 0 = \lambda_0 u_1 + \lambda_7/\tau \\ \frac{\partial \mathcal{H}}{\partial u_2} = 0 = \lambda_0 u_2 + \lambda_8/\tau \end{cases}$$

Using $\lambda_0 = 1$, $u_1 = -\lambda_7/\tau$ and $u_2 = -\lambda_8/\tau$, we obtain

$$\begin{cases} \dot{x}_1(t) = x_3(t) \\ \dot{x}_2(t) = x_4(t) \\ \mathrm{m}\dot{x}_3(t) = x_5(t) \\ \mathrm{m}\dot{x}_4(t) = x_6(t) \\ \tau\dot{x}_5(t) = -x_5(t) + x_7(t) \\ \tau\dot{x}_6(t) = -x_6(t) + x_8(t) \\ \tau\dot{x}_7(t) = -x_7(t) - \lambda_7(t)/\tau \\ \tau\dot{x}_8(t) = -x_8(t) - \lambda_8(t)/\tau \end{cases}$$

and

$$\begin{cases} \dot{\lambda}_1(t) = \frac{\partial \mathcal{H}}{\partial x_1} = 0 \\ \dot{\lambda}_2(t) = \frac{\partial \mathcal{H}}{\partial x_2} = 0 \\ \dot{\lambda}_3(t) = \frac{\partial \mathcal{H}}{\partial x_3} = -\lambda_1(t) \\ \dot{\lambda}_4(t) = \frac{\partial \mathcal{H}}{\partial x_4} = -\lambda_2(t) \\ \dot{\lambda}_5(t) = \frac{\partial \mathcal{H}}{\partial x_5} = -\lambda_3(t)/\mathrm{m} + \lambda_5(t)/\tau \\ \dot{\lambda}_6(t) = \frac{\partial \mathcal{H}}{\partial x_6} = -\lambda_4(t)/\mathrm{m} + \lambda_6(t)/\tau \\ \dot{\lambda}_7(t) = \frac{\partial \mathcal{H}}{\partial x_7} = -\lambda_5(t)/\tau + \lambda_7(t)/\tau \\ \dot{\lambda}_8(t) = \frac{\partial \mathcal{H}}{\partial x_8} = -\lambda_6(t)/\tau + \lambda_8(t)/\tau \end{cases}$$

where we have removed the variables x_0 and λ_0 which are not coupled to other variables. This is a linear system with boundary conditions on x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8. This system can be solved analytically. We note

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix} \text{ and } \boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \\ \lambda_8 \end{pmatrix}$$

We write

$$\begin{pmatrix} \boldsymbol{x}(t) \\ \boldsymbol{\lambda}(t) \end{pmatrix} = D(t)\boldsymbol{c} \tag{6}$$

where $D(t)$ is the 16×16 solution matrix, and $\boldsymbol{c}$ a 16×1 vector of constants to be obtained from boundary conditions. To obtain $\boldsymbol{c}$, we write

$$\begin{pmatrix} \boldsymbol{x}(t_0) \\ \boldsymbol{\lambda}(t_0) \end{pmatrix} = D(t_0)\boldsymbol{c}$$

and

$$\begin{pmatrix} \boldsymbol{x}(t_f) \\ \boldsymbol{\lambda}(t_f) \end{pmatrix} = D(t_f)\boldsymbol{c}$$

with

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0 = \begin{pmatrix} x_1^0 \\ x_2^0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } \boldsymbol{x}(t_f) = \boldsymbol{x}_f = \begin{pmatrix} x_1^f \\ x_2^f \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

but $\boldsymbol{\lambda}(t_0)$ and $\boldsymbol{\lambda}(t_f)$ are unknown. The trick is to write in block matrix notations

$$\begin{pmatrix} \boldsymbol{x}_0 \\ \boldsymbol{\lambda}(t_0) \end{pmatrix} = \begin{pmatrix} D_{11}(t_0) & D_{12}(t_0) \\ D_{21}(t_0) & D_{22}(t_0) \end{pmatrix} \begin{pmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \end{pmatrix}$$

where D_{11}, D_{12}, D_{21}, D_{22} are 8×8 matrices extracted from $D(t)$ (raws 1-8/columns 1-8, raws 1-8/columns 9-16, raws 9-16/columns 1-8, raws 9-16/columns 9-16), and $\boldsymbol{c}_1$, $\boldsymbol{c}_2$ 8-coordinates vectors extracted from $\boldsymbol{c}$ (raws 1-8, raws 9-16).

In the same way,

$$\begin{pmatrix} \boldsymbol{x}_f \\ \boldsymbol{\lambda}(t_f) \end{pmatrix} = \begin{pmatrix} D_{11}(t_f) & D_{12}(t_f) \\ D_{21}(t_f) & D_{22}(t_f) \end{pmatrix} \begin{pmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \end{pmatrix}$$

Thus

$$\begin{pmatrix} \boldsymbol{x}_0 \\ \boldsymbol{x}_f \end{pmatrix} = \begin{pmatrix} D_{11}(t_0) & D_{12}(t_0) \\ D_{11}(t_f) & D_{12}(t_f) \end{pmatrix} \boldsymbol{c}$$

which leads to

$$\boldsymbol{c} = \begin{pmatrix} D_{11}(t_0) & D_{12}(t_0) \\ D_{11}(t_f) & D_{12}(t_f) \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{x}_0 \\ \boldsymbol{x}_f \end{pmatrix}$$

Once $\boldsymbol{c}$ is known, the solution comes from Eq. 6. To calculate the controls, use $u_1 = -\lambda_7/\tau$ and $u_2 = -\lambda_8/\tau$, which corresponds to the two last lines of Eq. 6.

4 Optimal feedback control simulation of a visuomotor rotation

We consider a 2D mass actuated by second-order linear muscles (Eq. 5). We know how to set an optimal feedback control law for this object (Eqs. 6, 2 and 4). We define the observation process by the observation matrix

$$\mathbf{H} = \begin{pmatrix} \mathbf{I}_8 \\ \mathbf{I}_2 \end{pmatrix}$$

The upper part of the observation matrix corresponds to *proprioceptive* observation and the lower part to *visual* observation. When a visuomotor rotation (angle θ) is imposed, the observation matrix become

$$\mathbf{H}^\theta = \begin{pmatrix} \mathbf{I}_8 \\ \mathbf{R} \end{pmatrix}$$

where

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

The estimation process depends on the characteristics of noise on the dynamics and observation (Ω_{dyn} and Ω_{obs}). We set

$$\Omega_{\text{dyn}} = \sigma_{\text{dyn}} \mathbf{I}_8$$

and

$$\Omega_{\text{obs}} = \begin{pmatrix} \sigma_{\text{obs}}^{\text{prop}} \mathbf{I}_8 & \mathbf{0}_{8\times 2} \\ \mathbf{0}_{2\times 8} & \sigma_{\text{obs}}^{\text{vis}} \mathbf{I}_2 \end{pmatrix}$$

The ratio $\sigma_{\text{obs}}^{\text{prop}}/\sigma_{\text{obs}}^{\text{vis}}$ defines the relative influence of proprioceptive and visual observations.

Now we need to consider delays in the observation process. We note Δ_{prop} and Δ_{vis} the proprioceptive and visual delay, respectively. For simplicity, we assume that $\Delta_{\text{prop}} = \Delta_{\text{vis}} = \Delta$. We define $d = \Delta/\eta$. We rewrite the discretized dynamics (Eq. 3) as

$$\tilde{\boldsymbol{x}}_{t+1} = \tilde{\mathtt{A}}\tilde{\boldsymbol{x}}_t + \tilde{\mathtt{B}}\boldsymbol{u}_t$$

where the ˜ indicates an extended state representation. We define the extended state as

$$\tilde{\boldsymbol{x}}_t = \begin{pmatrix} \boldsymbol{x}_t \\ \mathbf{H}\boldsymbol{x}_{t-d} \\ \vdots \\ \vdots \\ \mathbf{H}\boldsymbol{x}_{t-1} \end{pmatrix},$$

the extended dynamic matrix as

$$\tilde{\mathtt{A}} = \begin{pmatrix} \mathtt{A} & \mathbf{0}_{8\times10} & \cdots & \cdots & \mathbf{0}_{8\times10} \\ \mathbf{0}_{10\times8} & \mathbf{0}_{10} & \mathbf{I}_{10} & \cdots & \mathbf{0}_{10} \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \mathbf{0}_{10\times8} & \mathbf{0}_{10} & \cdots & \mathbf{0}_{10} & \mathbf{I}_{10} \\ \mathbf{H} & \mathbf{0}_{10} & \cdots & \cdots & \mathbf{0}_{10} \end{pmatrix},$$

the extended control matrix as

$$\tilde{\mathtt{B}} = \begin{pmatrix} \mathtt{B} \\ \mathbf{0}_{10\times2} \\ \vdots \\ \vdots \\ \mathbf{0}_{10\times2} \end{pmatrix},$$

and the extended observation matrix as

$$\tilde{\mathbf{H}} = \begin{pmatrix} \mathbf{0}_{10\times8} & \mathbf{I}_{10} & \mathbf{0}_{10} & \cdots & \mathbf{0}_{10} \end{pmatrix}.$$

The problem can now be solved for the extended state representation.

5 Simulations

A simulation program was written in Python using `sympy` (link to the program). Parameters were: $m = 1$ kg, $t_f = 0.5$ s, $\sigma_{\text{dyn}} = 1$, $\sigma_{\text{obs}}^{\text{prop}} = 0.1$, $\sigma_{\text{obs}}^{\text{vis}} = 0.001$, $\Delta_{\text{prop}} = 0.1$ s, $\Delta_{\text{vis}} = 0.1$ s, , $t_f = 0.5$ s, $\eta = 0.01$ s. Movement amplitude was 0.1 m. We simulated three conditions: veridical visual feedback ($\theta = 0$ deg; Fig. 1), rotated visual feedback with an unadapted estimation process ($\theta = 45$ deg; Fig. 2), rotated visual feedback with an adapted estimation process ($\theta = 45$ deg; Fig. 3).

In the first condition, the hand trajectory and the visual feedback were straight to the target (Fig. 1). In the second condition, the hand trajectory was first directed toward the target, but after some delay deviated to compensate online for the perturbation (Fig. 2). In the third condition, the hand trajectory was straight to the target and the visual feedback were straight at θ deg from the target (Fig. 1).

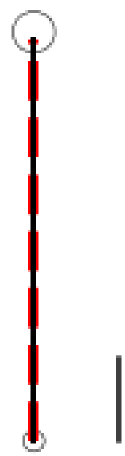

Figure 1: Simulation with veridical visual feedback. (*small circle*) starting point; (*large circle*) target; (*black*) visual feedback trajectory; (*dashed red*) actual hand trajectory. Scale: 2 cm.

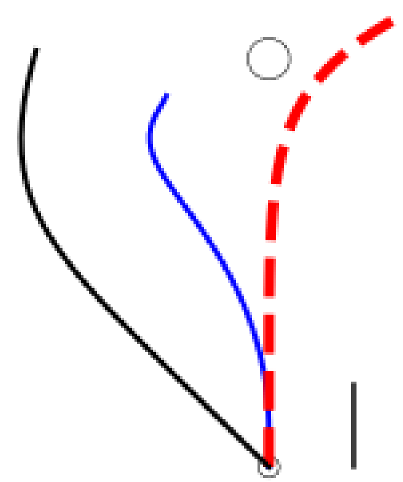

Figure 2: Simulation with rotated visual feedback when the estimation process is unaware of the perturbation. (*blue*) estimated hand trajectory.

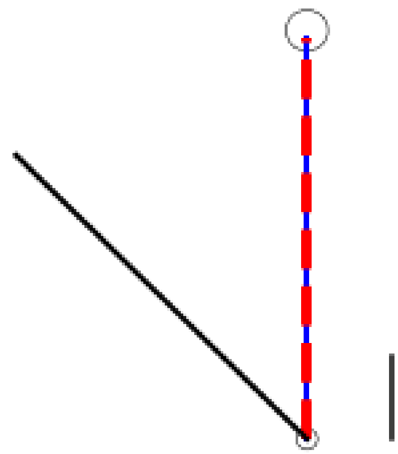

Figure 3: Simulation with rotated visual feedback when the estimation process is aware of the perturbation